LEVEL

# Systems Optimization Laboratory

Department of Operations Research
Stanford University
Stanford, CA 94305

81 10 14

SYSTEMS OPTIMIZATION LABORATORY
DEPARTMENT OF OPERATIONS RESEARCH
STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305

ON THE RELATIONSHIP BETWEEN THE HAUSDORFF DISTANCE
AND MATRIC DISTANCES OF ELLIPSOIDS

by

Alan J. Hoffman
Jean-Louis Goffin*

TECHNICAL REPORT SOL-81-10

August 1981

*IBM T.J. Watson Research Center, Yorktown Heights, N.Y., USA.

**McGill University, Montreal, Quebec, Canada.

# ABSTRACT

The space of ellipsoids may be metrized by the Hausdorff distance or by the sum of the distance between their centers and a distance between matrices. Various inequalities between metrics are established.

It implies that the square root of positive semidefinite symmetric matrices satisfies a Lipschitz condition, with a constant which depends only on the dimension of the space.

Key words

Ellipsoids, Hausdorff distance, matrices, square root matrix.

1. <u>Distance between ellipsoids as sets</u>.

Ellipsoids in $R^n$ may be viewed as elements of the set of subsets of $R^n$, subsets which could be restricted to be compact, convex and centrally symmetric. The set of subsets of $R^n$ is usually metrized by the Hausdorff metric [2]:

$$\delta(E,F) = \text{Max}\{\underset{y\in F}{\text{Sup}}\ \underset{x\in E}{\text{Inf}}\|x - y\|,\ \underset{x\in E}{\text{Sup}}\ \underset{y\in F}{\text{Inf}}\|x - y\|\}$$

$$= \text{Inf}\{\delta \geq 0 : E + \delta S \supset F,\ F + \delta S \supset E\}\ ,$$

where $E$ and $F$ are subsets of $R^n$, $\|\ \|$ represents the Euclidean norm, and $S = \{x \in R^n : \|x\| \leq 1\}$ is the unit ball.

If $E$ and $F$ are convex, then

$$\delta(E,F) = \text{Sup}\{|h(x,E) - h(x,F)| : \|x\| = 1\}\ ,$$

where $h(x,E) = \text{Sup}\{(x,y) : y \in E\}$ is the support function of $E$ (see Bonnensen-Fenchel [1]) and $(.,.)$ denotes the scalar product.

If $E$ and $F$ are convex and contain the origin in their interiors, then

$$\delta(E,F) = \text{Sup}\{|g(x,E^d) - g(x,F^d)| : \|x\| = 1\}$$

where $E^d = \{x \in R^n : (x,y) \leq 1\ \forall y \in E\}$ is the dual of $E$, and

$$g(x,E^d) = \text{Inf}\{\mu \geq 0 : x \in \mu E^d\}$$

is the distance function, or gauge, of $E^d$; this follows because $h(x,E) = g(x,E^d)$.

If $E$ and $F$ are convex, full-dimensional, and centrally symmetric with respect to the origin, then $E^d$ and $F^d$ inherit the same

properties, and $g(x,E^d)$ and $g(x,F^d)$ define norms on $R^n$. Thus $\delta(E,F)$ may be viewed as a distance between norms on $R^n$.

The Hausdorff distance is invariant under congruent, but not affine, transformations, and reduced by projection. It will be assumed throughout that the space of ellipsoids contains the degenerate ellipsoids. The space of ellipsoids is not closed under addition.

The following lemma indicates that it will be sufficient to study ellipsoids centered at the origin.

### Lemma 1

Let $E$ and $F$ be two subsets of $R^n$, compact, convex and symmetric with respect to the origin; let $E^1 = e + E$ and $F^1 = f + F$, then

$$\delta(E^1,F^1) \leq \delta(E,F) + \|e-f\| \leq 2\delta(E^1,F^1)$$

$$\delta(E,F) \leq \delta(E^1,F^1) \quad \|e-f\| \leq \delta(E^1,F^1) \ .$$

### Proof:

$$\delta(E^1,F^1) = \text{Sup}\{|h(x,E^1) - h(x,F^1)| : \|x\| = 1\}$$

$$= \text{Sup}\{|h(x,E) - h(x,F) + (e-f,x)| : \|x\| = 1\}$$

$$\leq \text{Sup}\{|h(x,E) - h(x,F)| : \|x\| = 1\} + \text{Sup}\{|(e-f,x)| : \|x\| = 1\}$$

$$= \delta(E,F) + \|e-f\| \ .$$

Conversely

$$-\delta(E^1,F^1) \leq h(x,E) - h(x,F) + (e-f,x) \leq \delta(E^1,F^1) \quad \forall \|x\| = 1 ;$$

now $h(x,E) = h(-x,E)$ and $h(x,F) = h(-x,F)$ as $E$ and $F$ are symmetric with respect to the origin, and thus

$$-\delta(E^1,F^1) \le h(x,E) - h(x,F) - (e-f,x) \le \delta(E^1,F^1) \quad \forall \|x\| = 1 \ .$$

Adding and subtracting, one gets:

$$-\delta(E^1,F^1) \le h(x,E) - h(x,F) \le \delta(E^1,F^1)$$

$$-\delta(E^1,F^1) \le (e-f,x) \le \delta(E^1,F^1) \quad \forall \|x\| = 1 \ ,$$

and hence $\delta(E,F) \le \delta(E^1,F^1)$ and $\|e-f\| \le \delta(E^1,F^1)$. QED.

## 2. Ellipsoids as vectors and matrices.

Ellipsoids may also be represented by a vector (its center) and a matrix (its size, shape and position):

$$E = e + \bar{A}S = \{x \in R^n : x = e + \bar{A}t, \ \|t\| = 1\} \ ;$$

note that $h(x,E) = (e,x) + \|\bar{A}^T x\|$. If $\bar{A}$ is nonsingular, then:

$$E = \{x \in R^n : (x-e)^T(\bar{A}\bar{A}^T)^{-1}(x-e) \le 1\} \ .$$

To any ellipsoid is associated an equivalence class of matrices; in fact $E = e + \bar{A}S = e + \bar{\bar{A}}S$ if and only if $\bar{\bar{A}} = \bar{A}O$ where $O$ is an orthogonal matrix, or equivalently if $\bar{A}\bar{A}^T = \bar{\bar{A}}\bar{\bar{A}}^T$. Define now $H = \bar{A}\bar{A}^T$, and $A = H^{1/2}$, then, in the remainder of this paper an ellipsoid will be defined by

$$E = e + AS = e + H^{1/2}S$$

where $A$ and $H$ are positive semidefinite symmetric matrices. Using

any of these two definitions there exists a one-to-one correspondence between ellipsoids and points $(e,A)$ in $R^n \times p(R^n)$ (respectively $(e,H) \in R^n \times p(R^n)$), where $p(R^n)$ is the set of $n \times n$ positive semidefinite symmetric matrices.

One could have tried to associate to an ellipsoid a lower triangular matrix $L(H = LL^T)$; $L$ is unique if $H$ is nonsingular, but not necessarily so if $H$ is singular. This is the key reason why the results of this paper will not extend to the case of Cholesky factors.

If $A$ is nonsingular, then

$$E = \{x \in R^n : (x-e)^T A^{-2} (x-e) \leq 1\}$$

$$= \{x \in R^n : (x-e)^T H^{-1} (x-e) \leq 1\} .$$

We may now define two matric distances on the space of ellipsoids.

Let $E = e + AS = e + H^{1/2} S$ and $F = f + BS = f + K^{1/2} S$ be two ellipsoids in $R^n$, where $A$, $B$, $H$ and $K$ are positive semidefinite symmetric matrices, then define:

$$d(E,F) = \|e-f\| + \|A-B\|$$

$$\Delta(E,F) = \|e-f\| + \|H-K\|^{1/2} = \|e-f\| + \|A^2-B^2\|^{1/2} ,$$

where $\| \|$, for matrices, is the spectral norm.

It is clear that $d$ and $\Delta$ satisfy the axioms for a metric (or distance).

Various inequalities between $d$, $\Delta$ and $\delta$ will be proved in the next section; the relationship between $d$ and $\delta$ is the closest one, as $d$ and $\delta$ are related by inequalities involving constants depending only upon the dimension of the space.

The inequalities imply that the three metrics define the same topology on the space of ellipsoids, but, more strongly, that rates of convergence can be related.

The inequalities between $d$ and $\delta$ imply that the rates of convergence of a sequence of ellipsoids may be studied within a space of sets, or a space of matrices, and that the two rates are identical.

3. Inequalities between distances.

If $E$ and $F$ are ellipsoids centered at the origin, and $E^1 = e + E$, $F^1 = f + F$, then

$$d(E^1, F^1) = \|e-f\| + d(E,F)$$

$$\Delta(E^1, F^1) = \|e-f\| + \Delta(E,F) \; ;$$

and Lemma 1 indicates that it is enough to study ellipsoids centered at the origin. In that case:

$$d(E,F) = \|A-B\|$$

$$\Delta(E,F) = \|H-K\|^{1/2} = \|A^2-B^2\|^{1/2}$$

$$\delta(E,F) = \mathrm{Sup}\{|\,\|Ax\| - \|Bx\|\,| : \|x\| = 1\} \; .$$

Theorem 2

Let $E = AS$ and $F = BS$ be two ellipsoids in $R^n$, centered at the origin, where $A$ and $B$ are $n \times n$ positive semidefinite symmetric matrices, then:

$$k_n^{-1}\|A-B\| \leq \mathrm{Sup}\{|\,\|Ax\| - \|Bx\|\,| : \|x\| = 1\} \leq \|A-B\|$$

or

$$\delta(E,F) \leq d(E,F) \leq k_n \delta(E,F)$$

where $k_n = 2\sqrt{2}n(n+2)$.

Proof:

For the first part, one has

$$\big| \|Ax\| - \|Bx\| \big| \leq \|Ax-Bx\| = \|(A-B)x\| \leq \|A-B\|\|x\| ,$$

and $\text{Sup}\{ \big| \|Ax\| - \|Bx\| \big| : \|x\| = 1 \} \leq \|A-B\|$.

For the second part, let $\delta = \text{Sup}\{ \big| \|Ax\| - \|Bx\| \big| : \|x\| = 1 \}$, and $\alpha_n \leq \alpha_{n-1} \leq \cdots \leq \alpha_1, \beta_n \leq \beta_{n-1} \leq \cdots \leq \beta_1$ be the ordered eigenvalues of $A$ and $B$ (clearly all real and nonnegative numbers).

The maximum characterization for the eigenvalues of Hermitian matrices gives:

$$\alpha_k^2 = \underset{S_k}{\text{Max}} \ \underset{x \in S_k}{\text{Min}} \ x^T A^2 x ,$$

where $S_k$ represents the intersection of any $k$ dimensional subspace with the unit spherical surface; assume that $S_k^*$ gives the maximum. Now define $x_k^*$ by

$$x_k^{*T} B^2 x_k^* = \underset{x \in S_k^*}{\text{Min}} \ x^T B^2 x .$$

Thus

$$\beta_k^2 = \underset{S_k}{\text{Max}} \ \underset{x \in S_k}{\text{Min}} \ x^T B^2 x \geq \underset{x \in S_k^*}{\text{Min}} \ x^T B^2 x = \|Bx_k^*\|^2$$

and

$$\alpha_k^2 = \min_{x \in S_k^*} x^T A^2 x \le |Ax_k^*|^2 \; ;$$

it follows that

$$\alpha_k - \beta_k \le |Ax_k^*| - |Bx_k^*| \le \delta \; .$$

Reversing the argument, $\beta_k - \alpha_k \le \delta$, and $|\alpha_k - \beta_k| \le \delta$ $\forall k = 1, \ldots, n.$

The content of the theorem is unchanged if $A$ is replaced by $O^T A O$ and $B$ by $O^T B O$, where $O$ is any orthogonal matrix; hence we may assume that $A$ is diagonal, and that

$$\alpha_i = a_{ii} \qquad \forall i = 1, \ldots, n \; .$$

Denote by $B_k = Be_k$ the $k^{th}$ column of $B$, where $e_k$ is the $k^{th}$ column of the identity matrix; then

$$|\alpha_k - |B_k|| = | |Ae_k| - |Be_k| | \le \delta \qquad \forall k = 1, \ldots, n \; .$$

Hence

$$|\beta_k - |B_k|| \le |\beta_k - \alpha_k| + |\alpha_k - |B_k|| \le 2\delta \qquad \forall k = 1, \ldots, n \; .$$

Now

$$|B_k|^2 = (\sum_{i=1}^{n} b_{ik}^2) \ge b_{kk}^2 \; ;$$

thus

$$0 \le |B_k| - b_{kk}$$

$$0 \leq \sum_{k=1}^{n} (\|B_k\| - b_{kk}) = \sum_{k=1}^{n} (\|B_k\| - \beta_k) \leq \sum_{k=1}^{n} |\|B_k\| - \beta_k| \leq 2n\delta \ ,$$

implying that

$$0 \leq \|B_k\| - b_{kk} \leq 2n\delta \qquad \forall k = 1, \ldots, n \ .$$

This leads to:

$$|\alpha_k - b_{kk}| \leq |\alpha_k - \beta_k| + |\beta_k - \|B_k\|| + |\|B_k\| - b_{kk}| \leq (2n+3)\delta \ .$$

Let $D = \text{Diag} (b_{kk})$, and $x$ be any vector of unit length, then:

$$|\|Bx\| - \|Dx\|| \leq |\|Bx\| - \|Ax\|| + |\|Ax\| - \|Dx\||$$

$$\leq \delta + \|A-D\| \leq \delta + (2n+3)\delta = (2n+4)\delta \ ,$$

as $\|A-D\| = \underset{k=1,\ldots,n}{\text{Max}} |\alpha_k - b_{kk}| \leq (2n+3)\delta.$

It remains to show that the off diagonal elements of $B$ are bounded by a multiple of $\delta$. If $b_{ii} = 0$, or $b_{kk} = 0$, then $b_{ik} = 0$ $(i \neq k)$ as $b_{ik}^2 \leq b_{ii}b_{kk}$ by the positive semidefiniteness of $B$. So assume that $b_{ii} > 0$, $b_{kk} > 0$, and let $a = b_{ii}$, $b = b_{kk}$, $c = |b_{ik}|$ and $\sigma = +1$ (resp. -1) if $b_{ik}$ is positive (resp. negative).

Choose $z = \dfrac{1}{\sqrt{a^2 + b^2}} (be_i + \sigma ae_k)$:

$$\|Bz\| = \frac{1}{\sqrt{a^2+b^2}} \|bB_i + \sigma aB_k\|$$

$$= \frac{1}{\sqrt{a^2+b^2}} \|(ab+ac)e_i + \sigma(ab+bc)e_k + \sum_{j \neq i,k} (bb_{ji} + \sigma ab_{jk})e_j\|$$

$$\geq \frac{1}{\sqrt{a^2+b^2}} \; \| (ab+ac)e_i + \sigma(ab+bc)e_k \|$$

$$= \frac{1}{\sqrt{a^2+b^2}} \; ((ab+ac)^2 + (ab+bc)^2)^{1/2}$$

$$= \left( c^2 + 2 \frac{abc(a+b)}{a^2 + b^2} + 2 \frac{a^2 b^2}{a^2 + b^2} \right)^{1/2}$$

$$= \left( \frac{\sqrt{2}ab}{\sqrt{a^2+b^2}} + d \right) ,$$

where this last equation defines $d$ $(d > 0)$.

Now, as $\| Dz \| = \frac{\sqrt{2}ab}{\sqrt{a^2+b^2}}$ , it follows that $d \leq \| Bz \| - \| Dz \| \leq (2n+4)\delta$.

The value of $d$ is given by the positive root of

$$d^2 + \frac{2\sqrt{2}ab}{\sqrt{a^2+b^2}} d = c^2 + \frac{2abc(a+b)}{a^2 + b^2} \; ;$$

the left-hand side increases with $d$ $(d \geq 0)$ and is less than the right-hand side for $d = 0$ and $d = c/\sqrt{2}$, implying that the value of $d$ is greater than $c/\sqrt{2}$, and

$$c < d\sqrt{2} \leq 2\sqrt{2}(n+2)\delta .$$

Thus $|b_{ik}| \leq 2\sqrt{2}(n+2)\delta$, $\forall i,k$, $i \neq k$, and

$$\| A-B \|^2 \leq Tr(A-B)^2$$

$$= \sum_k (a_{kk} - b_{kk})^2 + \sum_i \sum_{i \neq k} b_{ik}^2$$

$$\leq n(2n+3)^2 \delta^2 + n(n-1)8(n+2)^2 \delta^2$$

$$\leq 8n^2(n+2)^2 \delta^2 \; ;$$

hence $\|A-B\| \leq 2\sqrt{2}n(n+2)\delta.$   QED

The next result, which compares the distances $\delta$ and $\Delta$, uses an operator theory proof, and hence carries to infinite dimensional Hilbert spaces.

## Theorem 3

Let $E = H^{1/2}S$ and $F = K^{1/2}S$ be two ellipsoids in $R^n$, centered at the origin, where $H$ and $K$ are positive semidefinite symmetric matrices, then

$$\delta(E,F) \leq \|H-K\|^{1/2} \leq [\delta^2(E,F) + \delta(E,F)\text{Max}(D(E),D(F))]^{1/2} \ ,$$

where $\delta(E,F) = \text{Sup}\{|(x^T Hx)^{1/2} - (x^T Kx)^{1/2}| : \|x\| = 1\}$, $\Delta(E,F) = \|H-K\|^{1/2}$, and $D(E) = 2\|H\|^{1/2}$ is the diameter of $E$; it may also be written as:

$$\|H-K\|/[(\|H-K\| + \text{Max}(\|H\|,\|K\|))^{1/2} + (\text{Max}(\|H\|,\|K\|))^{1/2}]$$

$$\leq \delta(E,F) \leq \|H-K\|^{1/2} \ .$$

## Proof:

Let $\delta = \delta(E,F)$; thus:

$$(x^T Hx)^{1/2} - (x^T Kx)^{1/2} \leq \delta\|x\| \qquad \forall x \ ,$$

hence

$$x^T Hx \leq \delta^2\|x\|^2 + 2\delta\|x\|(x^T Kx)^{1/2} + x^T Kx \qquad \forall x \ ,$$

$$\leq \delta^2\|x\|^2 + x^T Kx + \varepsilon^{-1}\delta^2\|x\|^2 + \varepsilon(x^T Kx) \qquad \forall x, \forall \varepsilon > 0$$

$$= \delta^2(1+\varepsilon^{-1})\|x\|^2 + (1+\varepsilon)(x^T Kx) \qquad \forall x, \forall \varepsilon > 0 \ .$$

We have

$$x^T(H-K)x \leq x^T(\delta^2(1+\epsilon^{-1})I + \epsilon K)x \qquad \forall x, \forall \epsilon > 0 ,$$

and similarly, reversing the argument,

$$x^T(H-K)x \geq -x^T(\delta^2(1+\eta^{-1})I + \eta H)x \qquad \forall x, \forall \eta > 0 .$$

These two equations imply that

$$\|H-K\| \leq \text{Max}\{\epsilon\|K\| + \delta^2(1+\epsilon^{-1}), \eta\|H\| + \delta^2(1+\eta^{-1})\} \qquad \forall \epsilon > 0, \forall \eta > 0 ;$$

taking $\epsilon = \delta/\|H\|^{1/2}$ and $\eta = \delta/\|K\|^{1/2}$, one gets

$$\|H-K\| \leq \delta^2 + 2\delta \, \text{Max}(\|H\|^{1/2}, \|K\|^{1/2}) .$$

For the second part, let $\Delta^2 = \|H-K\|$, thus

$$\left| x^T(H-K)x \right| \leq \Delta^2\|x\|^2 \qquad \forall x ;$$

using the inequality

$$|a-b| \leq \sqrt{|a^2-b^2|} \qquad (a,b \geq 0)$$

one gets

$$\left| (x^T H x)^{1/2} - (x^T K x)^{1/2} \right| \leq \Delta\|x\| \qquad \forall x ,$$

and

$$\delta(E,F) = \text{Sup}\{\left| (x^T H x)^{1/2} - (x^T K x)^{1/2} \right| : \|x\| = 1\} \leq \Delta = \|H-K\|^{1/2} .$$

QED

Theorems 2 and 3 can be combined to give a relationship between the distances $d$ and $\Delta$, which is a statement about square roots of matrices.

## Theorem 4

Let $H$ and $K$ be two $n \times n$ positive semidefinite matrices, and $A = H^{1/2}$, $B = K^{1/2}$, then

$$\ell_n^{-1} \| A-B \| \leq \| H-K \|^{1/2} \leq [2 \| A-B \| \text{Max}( \| A \|, \| B \|) + \| A-B \|^2]^{1/2}$$

or

$$\| H-K \| / [( \| H-K \| + \text{Max}( \| H \|, \| K \|))^{1/2} + (\text{Max}( \| H \|, \| K \|))^{1/2}]$$

$$\leq \| A-B \| \leq \ell_n \| H-K \|^{1/2} ,$$

where $\ell_n = k_n = 2\sqrt{2} n(n+2)$.

This theorem means that the square root satisfies a Lipschitz condition on the cone of positive semidefinite matrices:

$$\| H^{1/2} - K^{1/2} \| \leq \ell_n \| H-K \|^{1/2} \quad \forall H,K \in p(R^n) ,$$

where the Lipschitz constant depends only upon the dimension of $R^n$; $\ell_n$ is bounded by a polynomial of degree 1 in the dimension of $p(R^n)$.

It is now a simple matter to extend Theorems 2, 3 and 4 to the case of ellipsoids not necessarily centered at the origin.

## Theorem 5

Let $E = e + AS = e + H^{1/2}S$ and $F = f + BS = f + K^{1/2}S$ be two ellipsoids in $R^n$, and $A$, $B$, $H$ and $K$ be $n \times n$ positive semidefinite symmetric matrices. Denote $\delta = \delta(E,F)$, $d = d(E,F)$, $\Delta = \Delta(E,F)$ and $M = \text{Max}( \| A \|, \| B \|) = \text{Max}( \| H \|^{1/2}, \| K \|^{1/2}) = \frac{1}{2} \text{Max}(D(E),D(F))$, then the following inequalities are satisfied:

$$(k_n+1)^{-1}d \leq \delta \leq d \leq (k_n+1)\delta$$

$$\ell_n^{-1}d \leq \Delta \leq (d^2+2dM)^{1/2}$$

$$\Delta^2/(\sqrt{\Delta^2 + M^2} + M) \leq d \leq \ell_n\Delta$$

$$\delta \leq \Delta \leq \delta + (\delta^2+2\delta M)^{1/2}$$

$$\Delta^2/(2(M+\Delta)) \leq \delta \leq \Delta$$

with $k_n = \ell_n = 2\sqrt{2}n(n+2)$.

Proof:

Let $\varepsilon = \|e-f\|$, and $\delta_o$, $d_o$, and $\Delta_o$ be the distances between
$E - e$ and $F - f$.

One has $d = d_o + \varepsilon$, $\Delta = \Delta_o + \varepsilon$ and, by Lemma 1, $\delta \leq \delta_o + \varepsilon$, $\varepsilon \leq \delta$
and $\delta_o \leq \delta$. Hence a slight difference appears in the proofs for the
various cases.

For instance, Theorem 4 implies

$$\Delta_o \leq (d_o^2 + 2d_o M)^{1/2} .$$

Hence $\Delta = \Delta_o + \varepsilon \leq (d_o^2 + 2d_o M)^{1/2} + \varepsilon$; the maximum of the right-hand
side (subject to $d_o \geq 0$, $\varepsilon \geq 0$ and $\varepsilon + d_o = d$) is attained for
$\varepsilon = 0$ and $d_o = d$, and thus

$$\Delta \leq (d^2 + 2dM)^{1/2}$$

or

$$d \geq \Delta^2/(\sqrt{\Delta^2 + M^2} + M) .$$

The equivalent result from Theorem 3 implies

$$\Delta_o \leq (\delta_o^2 + 2\delta_o M)^{1/2} ,$$

hence

$$\Delta = \Delta_o + \epsilon \leq (\delta_o^2 + 2\delta_o M)^{1/2} + \epsilon ;$$

and the maximum of the right-hand side subject to $\epsilon \leq \delta$ and $\delta_o \leq \delta$ is clearly attained for $\epsilon = \delta$ and $\delta_o = \delta$, and thus

$$\Delta \leq (\delta^2 + 2\delta M)^{1/2} + \delta$$

or

$$\delta \geq \frac{\Delta^2}{2(M+\delta)} .$$

The other cases follow similarly. QED

4. Conclusion.

Three metrics on the space of ellipsoids have been shown to be linked by various inequalities, and hence the induced topologies are identical. Not only is the notion of convergence unique, but rates of convergence can be related. Similar results clearly hold if the Euclidean norm is replaced by any of the $L_p$ norms.

If $k_n$ and $\ell_n$ were defined to be the smallest constants satisfying Theorems 2 and 4 (with $\ell_n \leq k_n$), it would be quite interesting to know whether, or not, they must depend on $n$, the dimension.

5.  <u>References</u>.

1)  T. Bonnensen and W. Fenchel, <u>Theorie der Konvexen Korper</u>,
    Chelsea, New York, New York, U.S.A., 1948.

2)  F. Hausdorff, <u>Set Theory</u>, 2nd ed., Chelsea, New York, New York,
    U.S.A., 1962.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

| REPORT DOCUMENTATION PAGE | | READ INSTRUCTIONS BEFORE COMPLETING FORM |
|---|---|---|
| 1. REPORT NUMBER<br>SOL 81-10 | 2. GOVT ACCESSION NO.<br>AD-A105880 | 3. RECIPIENT'S CATALOG NUMBER |
| 4. TITLE (and Subtitle)<br><br>On the Relationship Between the Hausdorff Distance and Matric Distances of Ellipsoids | | 5. TYPE OF REPORT & PERIOD COVERED<br>Technical Report |
| | | 6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s)<br>Alan J. Hoffman and Jean-Louis Goffin | | 8. CONTRACT OR GRANT NUMBER(s)<br>N00014-75-C-0267 |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS<br>Department of Operations Research - SOL<br>Stanford University<br>Stanford, CA 94305 | | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS<br><br>NR-047-143 |
| 11. CONTROLLING OFFICE NAME AND ADDRESS<br>Office of Naval Research - Dept. of the Navy<br>800 N. Quincy Street<br>Arlington, VA 22217 | | 12. REPORT DATE<br>August 1981 |
| | | 13. NUMBER OF PAGES<br>15 |
| 14. MONITORING AGENCY NAME & ADDRESS(If different from Controlling Office) | | 15. SECURITY CLASS. (of this report)<br><br>UNCLASSIFIED |
| | | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE |

16. DISTRIBUTION STATEMENT (of this Report)

This document has been approved for public release and sale;
its distribution is unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

ellipsoids
Hausdorff distance

matrices
square root matrix

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The space of ellipsoids may be metrized by the Hausdorff distance or by the sum of the distance between their centers and a distance between matrices. Various inequalities between metrics are established.

It implies that the square root of positive semidefinite symmetric matrices satisfies a Lipschitz condition, with a constant which depends only on the dimension of the space.

DD FORM 1473 EDITION OF 1 NOV 68 IS OBSOLETE
1 JAN 73